

DIAGONAL COMPLEXES

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ABSTRACT. Generalizing a construction of J.L. Harer’s paper “The virtual cohomological dimension of the mapping class group of an orientable surface”, we introduce and study diagonal complexes \mathcal{C} and \mathcal{B} related to a (possibly bordered) oriented surface F equipped with a number of labeled fixed points. Investigation of some natural forgetful maps combined with length assignment proves homotopy equivalence for some of the complexes, for the space of metric ribbon graphs $RG_{g,n}^{met}$, for the tautological S^1 -bundles L_i , and for a more sophisticated bundle whose fibers are homeomorphic to some surgery of the surface F . The latter is shown to incorporate all the tautological S^1 -bundles.

1. INTRODUCTION

We introduce and study complexes of pairwise non-intersecting curves on an oriented surface (called *diagonals*). Their endpoints belong (by definition) to some fixed set of labeled marked points (called *vertices*). On the one hand, the complexes generalize the *associahedron* (or *Stash-eff polytope*). On the other hand, they are directly related to the spaces of metric ribbon graphs.

Associahedron and cyclohedron, [14] and [2]. Assume that $n > 2$ is fixed. We say that two diagonals in a convex n -gon are *non-intersecting* if they intersect only at their endpoints (or do not intersect at all). Consider all collections of pairwise non-intersecting diagonals¹ in the n -gon. This set is partially ordered by reverse inclusion, and it was shown by John Milnor, that the poset is isomorphic to the face poset of some convex $(n - 3)$ -dimensional polytope As_n called *associahedron*.

In particular, the vertices of the associahedron As_n correspond to the triangulations of the n -gon, and the edges correspond to edge flips in which one of the diagonals is removed and replaced by a (uniquely

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¹It is important that the vertices of the polygon are labeled, and therefore we do not identify collections of diagonals that differ on a rotation.

defined) different diagonal. Single diagonals are in a bijection with facets of As_n , and the empty set corresponds to the entire As_n .

There exist many explicit constructions of the associahedron: as a special instance of secondary polytope, truncation of simplex, etc.

There exist also many ways to meaningfully generalize the associahedron. In the present paper, following [8] and [1], we consider one more way of generalization.

Centrally symmetric collections of diagonals in a $2n$ -gon give rise to a convex polygon called *cyclohedron*, or *Bott–Taubes polytope*. Its first definition is the compactification of the configuration space of n points on the circle. In the present paper cyclohedron arises as a diagonal complex in Example 5.

Both associahedron and cyclohedron fit into the family of *graph associahedra* [3], and also into the family of *generalized associahedra* [4].

Metric ribbon graphs. Ribbon graphs with labeled boundary components, see [10], appear in the context of the paper as duals to the (defined below) arrangements of diagonals on a closed surface.

A *ribbon graph* is a connected graph (possibly with multiple edges and loops) together with a cyclic ordering on the set of germs of edges incident to each vertex. Besides, we assume that each vertex of a ribbon graph has at least three incident germs of edges. A ribbon graph yields an oriented surface, whose genus is called the genus of the graph. A ribbon graph Γ becomes a metric ribbon graph after attaching a positive number l_i to each of its edges d_1, \dots, d_d . Thus isomorphic classes of ribbon graphs label the cells of the space of metric ribbon graphs $RG_{g,n}^{met}$. It is known due to Harer, Mumford, Thurston, and Penner that the space $RG_{g,n}^{met}$ of metric ribbon graphs with n boundary components and genus g can be identified with the decorated moduli space of complex curves of genus g with n distinct labeled marked points. The latter equals the product of the moduli space with the positive cone $\widetilde{\mathcal{M}}_{g,n} = \mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n$.

By definition (see [9]), the tautological complex line bundle on $\mathcal{M}_{g,n}$ has the cotangent space $T_{v_i}^*(C)$ at the marked point v_i as the fiber over $(C, v_1, \dots, v_n) \in \mathcal{M}_{g,n}$. The associated circle bundle on $RG_{g,n}^{met}$ has the i -th boundary component (considered as a metric circle) as the fiber over a point $(\Gamma, l_1, \dots, l_d) \in RG_{g,n}^{met}$.²

²A necessary reminder: by a circle bundle we mean a bundle whose fiber is an oriented circle; (isomorphic classes of) complex line bundles correspond bijectively to circle bundles.

Curve complexes. Curve complexes (or arc complexes) exist in the literature in different frameworks and settings. The original idea comes from W. Thurston [15] (or even much earlier, from his unpublished preprint). It was developed for the sake of study of the mapping class group of a surface. Oversimplifying, the basic idea is to take a (possibly bordered) surface with a finite set of labeled distinguished points, and to associate a complex with the ground set (that is, the set of vertices) equal to homotopy classes of either closed curves, or (depending on the setting) curves with endpoints in the distinguished set. Simplices correspond to non-intersecting representatives of the homotopy classes. The mapping class group has a natural subgroup acting on the complex, so it makes sense to take the quotient space.

An interesting part of the quotient complex corresponds to the curves that cut the surface into a number of disks. The latter is the subject of the present paper.

The existing literature on curve complexes is quite large, since the latter proved to be related to different areas: cluster algebras, low-dimensional dynamical systems, Teichmüller spaces, moduli spaces of punctured complex curves, and many others. Throughout the paper we mention J.L. Harer's paper [8], where the subject of the paper (the diagonal complex) together with barycentric subdivision appears for the first time. We also mention R.C. Penner's paper [12] with very similar construction, where he classifies all the cases when the complex is sphere homeomorphic, and N. Ivanov's survey [5] with an extension of Thurston's original ideas.

Main results of the paper. To single out the complexes that are studied in the present paper, we call them *diagonal complexes*.

With a surface F with a number of labeled marked points, we associate two (explicitly constructed) cell complexes: the complex \mathcal{C} and its barycentric subdivision \mathcal{B} (Section 2). These complexes for closed surfaces appeared in a slight disguise in J.L. Harer's paper [8]; however, for the sake of the completeness, we present here our construction which is appropriate for consequent paragraphs.

If the surface F has no boundary, and under some other condition of stability, \mathcal{B} (as well as \mathcal{C}) is homotopy equivalent to $RG_{g,n}^{met}$. Moreover, in this case \mathcal{B} is a subcomplex of barycentric subdivision of $RG_{g,n}^{met}$ (Section 3). This result is also contained in [8].

The barycentric subdivision construction makes possible to attach lengths (some positive numbers) to the diagonals. This turns the complex \mathcal{B} (and therefore, also \mathcal{C}) to the space of metric arrangements, see Theorem 2. With these preparations at hand, we proved the following.

The homotopy type of \mathcal{B} (as well as \mathcal{C}) does not depend on the number n_i of points on a boundary component, provided that $n_i > 0$ (Section 4).

Contraction of a boundary component B_i induces a natural forgetful map $\mathcal{B}(F) \rightarrow \mathcal{B}(\overline{F})$ which is shown to be isomorphic to the tautological S^1 -bundle L_i (here \overline{F} is the surface F with contracted B_i) (Section 5). If \overline{F} is a closed surface, the tautological bundle is the classical one studied in [9].

A more sophisticated bundle appears if we allow *free* boundary components, that is, components with no fixed points. Contraction of a free component corresponds to a bundle whose fiber is some surface \overline{F} obtained from F by some surgery, see Section 6. This bundle is not trivial: in a sense, it incorporates all the tautological bundles related to the surface.

For a small example (one free boundary component in an n -gon) the associated complex \mathcal{C} has the combinatorics of cyclohedron.

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2. MAIN CONSTRUCTION AND INTRODUCTORY EXAMPLES

Assume that we have a surface F of genus g with b labeled boundary components B_1, \dots, B_b . We fix n distinct labeled points on F not lying on the boundary (these are called *free vertices*), and for each $i = 1, \dots, b$ we fix $n_i > 0$ distinct labeled points on the boundary component B_i . We assume that F can be triangulated with vertices at the marked points, that is, we exclude all "small" cases (like sphere with two marked points).

All these fixed points are called *vertices* of F . Altogether we have $N = n + \sum_{i=1}^b n_i$ vertices. The vertices that lie on the boundary split the boundary components into *edges*.

A *pure diffeomorphism* $F \rightarrow F$ is an orientation preserving bijective diffeomorphism which maps fixed points to fixed points and preserves the labeling. A pure diffeomorphism maps each boundary component to itself.

A *diagonal* is a simple (that is, not self-intersecting) smooth curve d on F whose endpoints are some of the (possibly one and the same) vertices such that

- (1) d does not intersect the boundary (except for its endpoints),

- (2) d is homotopic³ to no edge of the boundary.
- (3) d is non-contractible.
- (4) d contains no vertices (except for the endpoints).

An *amissible diagonal arrangement* (or an *admissible arrangement*, for short) is a non-empty collection of diagonals $\{d_j\}$ with the properties:

- (1) Each free vertex is an endpoint of some diagonal.
- (2) No two diagonals intersect (except for their endpoints).
- (3) No two diagonals are homotopic.
- (4) The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of open disks.

Definition 1. Two arrangements A_1 and A_2 are equivalent whenever there exists a pure diffeomorphism of F which maps bijectively A_1 to A_2 .

Remark. If there are no boundary components, equivalence classes of admissible arrangements correspond bijectively to ribbon graphs from $RG_{g,n}$.

Arrangements with maximal possible number of diagonals $Max = 6g + 3b + 2n + N - 6$ correspond to triangulations of F with vertices at fixed points. Here by triangulation we mean that the disks of the complement are combinatorial triangles, but possibly self-intersecting on the boundary. Arrangements with minimal possible number of diagonals $Min = 2g + n + b - 1$ have a unique disc in the complement.

A triple (g, b, n) is *stable* if no admissible arrangement has a non-trivial diffeomorphism (that is, each diffeomorphism maps each germ of each of d_i to itself). Triples with $b > 1$ are stable since a boundary component allows to set a linear ordering on the germs of diagonals emanating from each of its vertices. It is known⁴ that any triple with $n > 2g + 2$ is stable.

Throughout the paper we assume that **all the triples are stable**.

We introduce two posets of equivalence classes of admissible diagonal arrangements. In the sequel instead of saying "equivalence class of admissible diagonal arrangement" we say "admissible arrangement" for short.

³Here and in the sequel, we mean homotopy with fixed endpoints in the complement of the vertices $F \setminus Vert$. In other words, a homotopy never hits a vertex.

⁴This follows from Lefschetz fixed point theorem, as explained by Bruno Joyal in personal communications.

Poset C . Admissible arrangements are partially ordered by reversed inclusion: we say that $A_1 \leq A_2$ if there exists a pure diffeomorphism of F which maps A_2 in A_1 . If this is the case we say that the diffeomorphism *embeds* A_2 in A_1 . Such embeddings may be essentially different for one and the same pair of arrangements, see Examples 3 and 4.

Thus for the data $(g, b, n; n_1, \dots, n_b)$ we have a poset of all admissible arrangements $C_{g,b,n;n_1,\dots,n_b}$.

Example 1. *The poset $C_{0,1,0;n_1}$ is isomorphic to the face poset of the associahedron As_{n_1} . In this case any collection of pairwise non-intersecting diagonals is admissible.*

In view of this example we are going to generalize associahedron. The surface F plays the role of the polygon, marked points play the role of vertices.

Poset B . Each element of the poset $B_{g,b,n;n_1,\dots,n_b}$ is some admissible arrangement $A = \{d_1, \dots, d_m\}$ with a linearly ordered partition $\bigsqcup S_i = A$ into some non-empty sets S_i such that the first set in the partition S_1 is an admissible arrangement.

The partial order on B is generated by the following rule:

$(S_1, \dots, S_p) \leq (S'_1, \dots, S'_{p'})$ whenever one of the two conditions holds:

- (1) We have one and the same arrangement A , and $(S'_1, \dots, S'_{p'})$ is an order preserving refinement of (S_1, \dots, S_p) .
- (2) For all $i = 1, 2, \dots, p$, we have $S_i = S'_i$.

In other words, given (S_1, \dots, S_p) , to list all the elements of B that are smaller than (S_1, \dots, S_p) one has (1) to eliminate some (but not all!) of S_i from the end of the string, and (2) to replace some consecutive collections of sets by their unions.

Examples:

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3, d_1, d_6\}, \{d_4, d_7\}).$$

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}).$$

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7, d_8\}).$$

Minimal elements of B correspond to admissible arrangements. Maximal elements correspond to maximal arrangements A together with some minimal admissible subarrangement $A' \subset A$ and a linear ordering on the set $A \setminus A'$. For maximal elements, the number of sets in the partition $p = \text{Max} - \text{Min} + 1$.

Note that although A' may embed in A in different ways, the elements of B "know" the embedding:

Lemma 1. *If $(S_1, \dots, S_r) \leq (S'_1, \dots, S'_{r'})$ then there exists a unique (up to isotopy) order-preserving pure diffeomorphism of F which embeds $A = S_1 \cup \dots \cup S_r$ in $A' = S'_1 \cup \dots \cup S'_{r'}$.*

Proof. If $S_1 = S'_1$, the arrangement S_1 maps identically to itself since it has no automorphisms by stability assumption. The rest of the diagonals are diagonals in polygons, and are uniquely defined by their endpoints. Assume that $S_1 \subset S'_1$. For the rest of the cases it suffices to take $A = S_1$, $A' = A = S'_1 \sqcup S'_2$. If A embeds in A' in different ways, then A has a non-trivial isomorphism, which contradicts stability assumption. \square

Diagonal complexes: main construction.

For a fixed data $(g, b, n; n_1, \dots, n_b)$, we describe below a cell complex $\mathcal{C} = \mathcal{C}_{g,b,n;n_1,\dots,n_b}$ which is combinatorially isomorphic to the above described poset $C_{g,b,n;n_1,\dots,n_b}$. Its barycentric subdivision $\mathcal{B} = \mathcal{B}_{g,b,n;n_1,\dots,n_b}$ (which we prove to be well-defined) is combinatorially isomorphic to the above described poset $B_{g,b,n;n_1,\dots,n_b}$.

A cell complex K is *regular* if each k -dimensional cell c is attached to some subcomplex of the $(k-1)$ -skeleton of K via a bijective mapping on ∂c . It is important that \mathcal{C} is not a regular cell complex, and therefore is not uniquely defined by the poset of its cells. So to correctly present \mathcal{C} , we need to explicitly describe the cell attaching rules.

We construct $\mathcal{C}_{g,b,n;n_1,\dots,n_b}$ and $\mathcal{B}_{g,b,n;n_1,\dots,n_b}$ inductively, starting with zero-dimensional cells of \mathcal{C} . In \mathcal{C} , the latter correspond to arrangements with maximal possible number of diagonals, that is, to triangulations of F . In \mathcal{B} , they correspond to the same arrangements together with a one-element partition (namely, with $S_1 = A$). Note that they do not exhaust all minimal elements of \mathcal{B} .

Assume that the m -skeleton of \mathcal{C} is already constructed and is already barycentrically subdivided. Its cells in \mathcal{C} correspond to arrangements with at least $Max - m$ diagonals. Its cells in \mathcal{B} (that barycentrically subdivide cells in the m -skeleton of \mathcal{C}) correspond to (S_1, \dots, S_k) such that $k \leq m+1$, S_1 is an admissible arrangement, and $|S_1| \geq Max - m$.

Now we attach m -dimensional cells of \mathcal{C} one by one. We fix the barycentric subdivision of a cell in advance, before attaching it, so the new cells of the complex \mathcal{B} appears automatically.

Let us take a diagonal arrangement A containing $M = Max - m - 1$ diagonals.

All arrangements that strictly contain an embedding of A have already corresponding cells in the m -skeleton.

The complement $F \setminus A$ splits into a number of polygons, say, P_1, \dots, P_l with number of vertices v_1, \dots, v_l respectively. Adding a diagonal to A

means adding a diagonal to one of the polygons. Consider the Cartesian product of associahedra $P = \prod_{i=1}^l As_{v_i}$. Each facet⁵ of P is associated to adding a diagonal to one of the polygons. Each face of P is associated to adding a number of pairwise non-intersecting diagonals to some of polygons. Now take the barycentric subdivision of P . Since simplices in the barycentric subdivision of any polytope are encoded by nested sequences of its faces, in our case, simplices are labeled by (S_1, S_2, \dots, S_r) with $S_1 \supseteq A$. Moreover, their incidence relations are exactly the same as in B .

Example 2. *The center of P is labeled by $(S_1) = (A)$, centers of facets are labeled by $(A \cup \{d\})$, where d is a diagonal of some P_i . A segment that connects the center of P with the center of a facet is labeled by $(A, \{d\})$.*

Consider now all the simplices with $|S_1| \geq \text{Max} - m$ (or, equivalently, with $S_1 \neq A$). On the one hand, in the barycentric subdivision of the polytope P they form the boundary of P . On the other hand, they have already their representatives in the m -skeleton. May occur that two simplices have one and the same representative. Important is that the incidence relations in the skeleton and in ∂P are one and the same, so we may patch P to the skeleton along its boundary.

Proposition 1. *The cell complex \mathcal{B} is regular, \mathcal{C} is not.*

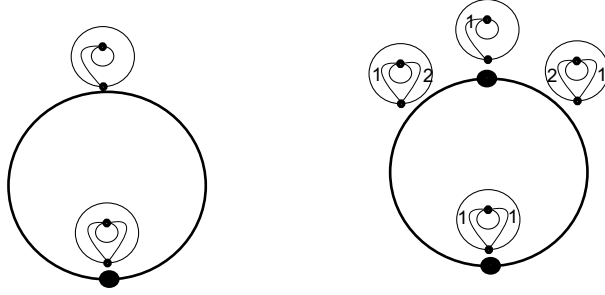
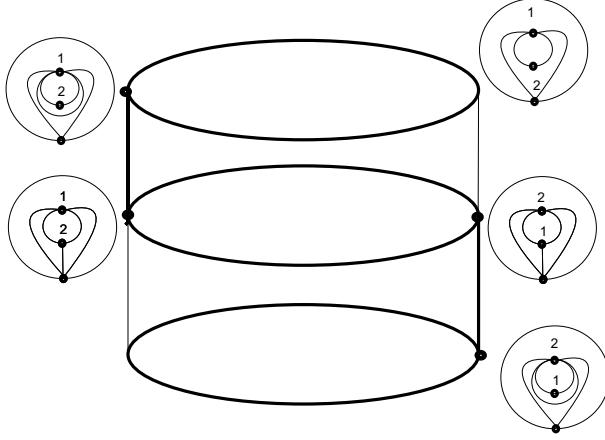
Proof. All the vertices of a simplex in \mathcal{B} correspond to different arrangements: they have different number of diagonals. \square

Example 3. $\mathcal{C}_{0,2,0;1,1}$ is a combinatorial circle. It has one vertex and one edge. $\mathcal{B}_{0,2,0;1,1}$, which is also a combinatorial circle, has two vertices and two edges, see Fig 1.

Example 4. $\mathcal{C}_{0,2,0;1,2}$ is the cylinder $I \times S^1$. It has four vertices, six edges, and two pentagonal cells, see Fig 2. Each of the pentagonal cells patches to itself by an edge.

Remark. Following [8], one defines another type of equivalence of arrangements (not as in Definition) 1: two arrangements A_1 and A_2 are *equivalent* whenever there exists a pure homotopy of F (that is, a homotopy that is identical on the boundary) which takes A_1 to A_2 . For this equivalence, similar construction as below is valid, and one arrives

⁵that is, a face of codimension one.


 FIGURE 1. Complexes $\mathcal{C}_{0,2,0;1,1}$ and $\mathcal{B}_{0,2,0;1,1}$

 FIGURE 2. Complex $\mathcal{C}_{0,2,0;2,1}$. We depict labels of all the vertices and one of the edges. This figure (without labels) appeared in [1]

to an (infinite) complex \mathcal{T} . Then \mathcal{C} equals $\mathcal{T}/PMC(F)$, where PMC is the *pure mapping class group*⁶ (or *pure modular group*) of F .

Remark. In the section we have exploited the (not new) idea of the *order complex* [16] of a poset. However, we stress that B is not the order poset of \mathcal{C} .

3. ATTACHING LENGTHS. RELATION TO $RG_{g,n}^{met}$

Admissible arrangements bijectively correspond to ribbon graphs by graph duality. Contracting an edge in a ribbon graph corresponds to eliminating a diagonal from an arrangement.

⁶That is, the group of isotopic classes of orientation-preserving diffeomorphisms $F \rightarrow F$ that preserve fixed points and the labeling.

Theorem 1. (1) $RG_{g,n}^{met}$ (considered as a topological space) is homotopy equivalent to $\mathcal{B}_{g,o,n}$ (and therefore also to $\mathcal{C}_{g,o,n}$).
 (2) $\mathcal{B}_{g,o,n}$ embeds in (the analog of)⁷ barycentric subdivision of $RG_{g,n}^{met}$.

The rough idea of the proof is to compare barycentric subdivision.

By definition, see [10], $RG_{g,n}^{met}$ is patched of the open balls. Each of the balls D^{m-1} correspond to some admissible arrangement of m diagonals. Let us fix some arrangement and enumerate its diagonals d_1, \dots, d_m . The corresponding open ball is the simplex Δ in \mathbb{R}^m defined by $\sum_{i=1}^m l_i = 1$, $l_i > 0$. Its closure $\bar{\Delta}$ is defined by $\sum_{i=1}^m l_i = 1$, $l_i \geq 0$. Each facet of Δ corresponds to removing a diagonal from the arrangement. If the arrangement remains admissible, the facet appears in $RG_{g,n}^{met}$, otherwise not.

Metric simplices of the barycentric subdivision $Bar(\bar{\Delta})$ are labeled by linearly ordered partitions of non-empty subsets of $[m] = \{1, \dots, m\}$. For instance, for $m = 6$ the label $(\{1, 3, 5\}\{2\})$ corresponds to the simplex $\sum l_i = 1$, $l_1 = l_3 = l_5 > l_2 > l_4 = l_6 = 0$. The vertices of a simplex (S_1, S_2, \dots, S_r) are labeled by (S_1) , $(S_1 \cup S_2)$, $(S_1 \cup S_2 \cup S_3)$, etc.

Facets of the simplex (defined by $l_i = 0$) appear in $RG_{g,n}^{met}$ if and only if removal of the diagonal d_i leaves an admissible arrangement. Therefore, the closure of a simplex with label (S_1, S_2, \dots, S_r) lies in $RG_{g,n}^{met}$ if and only if $\{d_i\}_{i \in S_1}$ is an admissible arrangement.

A simplex of $Bar(\bar{\Delta})$ is *weak* if some of its faces do not lie in $RG_{g,n}^{met}$, that is, iff S_1 is not admissible. Otherwise a simplex is *strong*. For a weak simplex σ of $Bar(\bar{\Delta})$ the set of its faces that do not appear in $RG_{g,n}^{met}$ form a contractible subset. Indeed, let σ be labeled by (S_1, \dots, S_r) . Its faces that are strong are labeled by $(S_1 \cup S_2 \cup \dots \cup S_q \cup *, *)$, where q is the smallest number that makes $S_1 \cup S_2 \cup \dots \cup S_q$ admissible, with no restriction on $*$. So a weak simplex can be contracted in $RG_{g,n}^{met}$.

So we contract all the weak simplices (or just eliminate all the open weak simplices) in $RG_{g,n}^{met}$ for all the cells of $RG_{g,n}^{met}$. We get a homotopy equivalent space patched of (closed) strong simplices. The patching rules are the same as in $\mathcal{B}_{g,o,n}$. \square

In view of the above construction, it is possible to attach lengths to diagonals for any complex \mathcal{B} , even when F has boundary components. This gives a *metric arrangement*. Namely, we have:

⁷ Although $RG_{g,n}^{met}$ is not a cell complex in the sense of A. Hatcher's book [7], one can barycentrically subdivide the open balls ("cells") it is patched of.

Theorem 2. *The support of $\mathcal{B}_{g,b,n;n_1,\dots,n_b}$ equals the space of admissible arrangements equipped with a length function*

$$l : \{d_i\} \rightarrow \mathbb{R}_{>0},$$

which satisfies the two conditions:

- (1) *For each (A, l) the length function l attains its maximal value on some admissible $A' \subseteq A$.*
- (2) $\sum_{d_i \in A} l(d_i) = 1$.

Vanishing of $l(d_i)$ means eliminating the diagonal d_i . □

Note that in our setting, no length is attached to the edges of F .

Definition 2. *Let v_i be a free vertex. For a metric diagonal arrangement let l_1, l_2, \dots, l_m be the lengths of diagonals emanating from v_i ⁸ coming in the counter clockwise order. The tautological circle bundle L_i on \mathcal{B} is the bundle whose fiber over a metric arrangement is the (combinatorial) polygon with consecutive edge lengths l_1, l_2, \dots, l_m .*

If there are no boundary components, L_i equals the tautological bundle introduced in [9].

Remark. One can relax the condition $\sum l_i = 1$ and define the length assignment to an arrangement as a point in the real projective space. This will be convenient in the subsequent sections where we consider erasing of some of diagonals.

Remark. Although Theorem 2 represents each simplex of the complex \mathcal{B} as some metric simplex, for the consequent paragraphs a reader may imagine each (combinatorial) simplex in $\mathcal{B}_{g,b,n;n_1,\dots,n_b}$ as a (Euclidean) equilateral simplex and to define the *support*, or *geometric realization* of the complex $|\mathcal{B}|_{g,b,n;n_1,\dots,n_b} = |\mathcal{C}|_{g,b,n;n_1,\dots,n_b}$ as the patch of these simplices.

4. CONTRACTION OF EDGES

Theorem 3. *Homotopy type of $|\mathcal{C}|_{g,b,n;n_1,\dots,n_b} = |\mathcal{B}|_{g,b,n;n_1,\dots,n_b}$ depends only on the triple (g, b, n) .*

Proof. Prove that $|\mathcal{C}|_{g,b,n;n_1,\dots,n_b}$ is homotopy equivalent to $|\mathcal{C}|_{g,b,n_1-1,\dots,n_b}$ provided that $n_1 > 1$. More precisely, we will show that $|\mathcal{B}|_{g,b,n;n_1-1,\dots,n_b}$ is a deformation retract of $|\mathcal{B}|_{g,b,n;n_1,\dots,n_b}$.

Choose an edge e with endpoints v', v'' (taken in counter-clockwise order) on the boundary component B_1 . Consider the following forgetful poset epimorphism (depending on the chosen edge).

⁸one and the same diagonal may appear twice.

$$\pi : B_{g,b,n;n_1,\dots,n_b} \rightarrow B_{g,b,n;n_1-1,\dots,n_b}$$

The defining rule is as follows. Take an element of $B_{g,b,n;n_1,\dots,n_b}$, it gives us some admissible arrangement together with a partition (S_1, \dots, S_r) . Contract the edge e ; it turns to a vertex v , which replaces the former vertices v', v'' . We obtain a (new) collection of diagonals related to the surface with contracted edge. Some of the diagonals may become either contractible or homotopic to an edge of F . Eliminate them. Some of the diagonals may become pairwise homotopy equivalent. In each class we leave exactly one that belongs to S_i with the smallest index i . Eventually some of the sets S_i may become empty in the process. Eliminate all the empty sets keeping the order of the rest. We obtain an element from $B_{g,b,n;n_1-1,\dots,n_b}$. It is easy to check that $A < A'$ implies $\pi(A) \leq \pi(A')$, so the map is indeed a poset morphism.

The poset morphism extends to a (uniquely defined) piecewise linear map

$$\pi : |\mathcal{B}|_{g,b,n;n_1,\dots,n_b} \rightarrow |\mathcal{B}|_{g,b,n;n_1-1,\dots,n_b}$$

which is linear on each of the simplices.

We denote it by the same letter π . The preimage of each point carries the structure of a regular cell complex; let us show that it is a combinatorial segment.

Take an inner point $x \in \sigma^{r-1}$ from $\mathcal{B}_{g,b,n;n_1-1,\dots,n_b}$ labeled by (S_1, \dots, S_r) . Assume that for σ^{r-1} , in the corresponding arrangement v has m emanating germs of diagonals and two germs of incident boundary edges $d_0, d_1, \dots, d_m, d_{m+1}$, coming in the counterclockwise order. Some of the germs may correspond to one and the same diagonal (edge). Each simplex in the preimage of x is obtained in the following way.

Expand the edge e either before all the germs, or after all the germs, or between between two consecutive germs. Now we have two cases:

- (1) Leave the arrangement as it is (keeping the partition). Then the arrangement corresponds to some one-dimensional simplex in the preimage.
- (2) Add a diagonal to the arrangement. Then the arrangement corresponds to a vertex of the preimage. Here we again have two cases:
 - (a) The new diagonal d' becomes homotopic to some $d \in S_i$ after collapsing e . Then we put d' in the same set S_i or to the right of it. We also can create a separate singleton $\{d'\}$ and put it to the right of S_i .

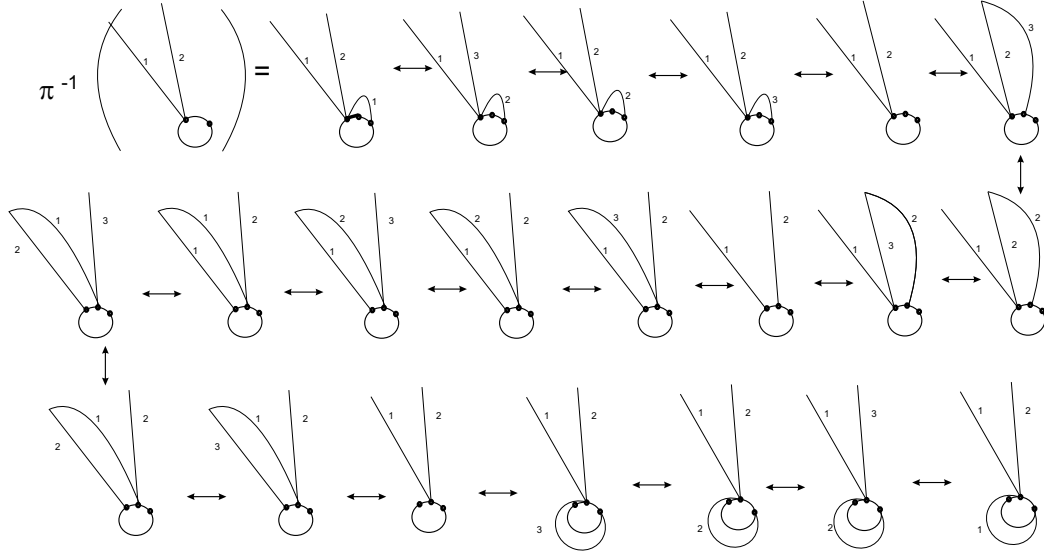


FIGURE 3. The preimage of a configuration is a combinatorial segment, and thus is contractible on the lefthand end of the segment. The edge which gets contracted is marked bold.

- (b) The new diagonal d' is homotopic to a piece of the boundary component B_1 . In this case we put d' in any of S_i , or create a separate singleton, see Figure 3).

One can check that altogether we have a segment in the preimage: first check that each vertex is incident to at most two segments, then check the connectivity of the preimage.

All of the simplices in the sequence are evidently distinct (one can easily show it by comparing degrees of vertices and capacities of sets S_i). So we indeed have a combinatorial segment.

Take a subcomplex $\mathcal{K} \subset \mathcal{B}_{g,b,n;n_1,\dots,n_b}$ labeled by all the arrangements which have a diagonal homotopic to the conjunction of e and the next edge of boundary in the clockwise order e' (such a curve is indeed a diagonal) lying in the set S_1 . Obviously such arrangement has no germs emanating from v' . An example of such arrangement is the leftmost in the Figure 3, that is, \mathcal{K} consists of all lefthandside vertices of the segments $\pi^{-1}(x)$, where x ranges over $|\mathcal{B}|_{g,b,n;n_1-1,\dots,n_b}$. Observe that π maps $|\mathcal{K}|$ isomorphically to $|\mathcal{B}|_{g,b,n;n_1-1,\dots,n_b}$.

Now we can fiberwisely retract $|\mathcal{B}|_{g,b,n;n_1,\dots,n_b}$ onto $|\mathcal{K}|$. □

5. CONTRACTION OF BOUNDARY COMPONENTS

Theorem 4. $\mathcal{C}_{g,k,n;1,n_2,n_3,\dots,n_b}$ (together with the below defined projection) is the tautological circle bundle L_1 over $\mathcal{C}_{g,b-1,n+1;n_2,n_3,\dots,n_b}$.

Proof. Consider the forgetful mapping

$$\pi : \mathcal{B}_{g,b,n;1,n_2,n_3,\dots,n_b} \rightarrow \mathcal{B}_{g,b-1,n+1;n_2,\dots,n_b}$$

The defining rule is literally the same as in the previous section, namely: take a simplex in $\mathcal{B}_{g,b,n;1,\dots,n_b}$. It corresponds to some admissible arrangement. After contraction, the boundary component B_1 becomes a free fixed point v . Some of the diagonals may become contractible. Eliminate them. Some of the diagonals may become homotopy equivalent. In each class we leave exactly one that belongs to S_i with the smallest index i .

The mapping induces a continuous mapping for the supports:

$$\pi : |\mathcal{B}|_{g,b,n;1,n_2,n_3,\dots,n_b} \rightarrow |\mathcal{B}|_{g,b-1,n;n_2,\dots,n_b}$$

Take a simplex σ from $\mathcal{B}_{g,b-1,n+1;n_2,\dots,n_b}$ labeled by (S_1, \dots, S_r) . Assume that for σ , in the corresponding arrangement v has m emanating germs of diagonals d_1, \dots, d_m , coming in the counterclockwise order. Some of the germs may correspond to one and the same diagonal. Consider the preimage of a point x lying in the interior of σ . The preimage carries the structure of a regular cell complex. The simplices in the preimage are obtained by the following procedure: place the new boundary component to the vertex v . Either leave it as it is, or add a curve which duplicates one of the two neighbor diagonals $d \in S_i$. Put the new diagonal d' either in the same set S_i as d , or to any of the sets with bigger indices, or as a singleton to any place to the right of S_i .

It is easy to see that $\pi^{-1}(x)$ is a combinatorial circle. Figure 4 depicts the preimage $\pi^{-1}(x)$ for the case when the collapsed boundary component has exactly two emanating diagonals, one from S_1 , and the other from S_2 .

The generic case is captured by the following observation: each preimage is connected, it carries a structure of one-dimensional cell complex, each vertex of which has exactly two adjacent edges.

We are almost done; however, we need a metric combinatorial circle to ensure that we have the tautological circle bundle. Let us take the combinatorial dual of the circle. Some of its vertices are labeled by placements of the new boundary component (with no duplication of diagonals, see Fig. 5). Let us call them *bold*. Eliminate all the other vertices by taking union of edges lying between bold vertices. Each

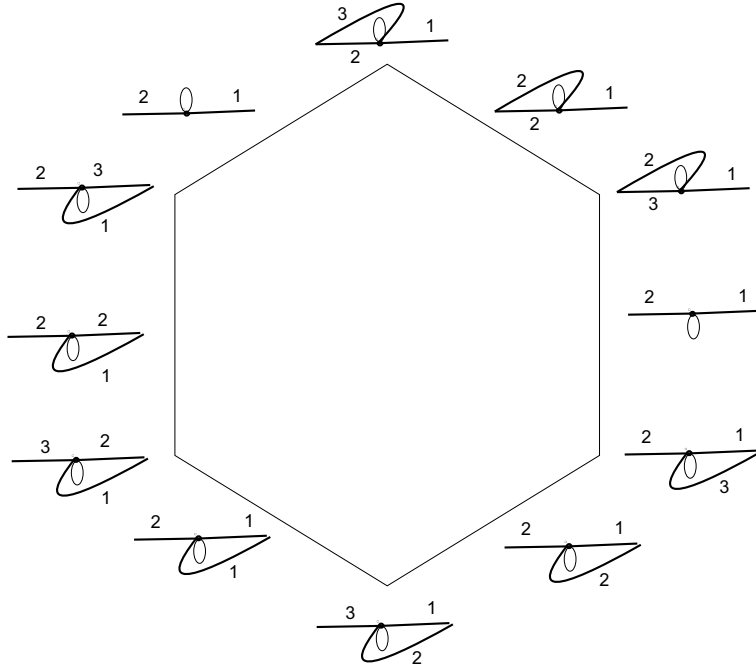


FIGURE 4. The preimage is the combinatorial hexagon. We depict here the corresponding arrangements locally, near the first boundary component, since the rest of the diagonals remains unchanged.

edge e of the new polygon is associated to some diagonal d emanating from v . Assign to the edge d the length $l(d)$.

The behaviour of the metric circle $\pi^{-1}(x)$ when the point x moves on the base is captured by the following observations:

If x stays in one and the same simplex of \mathcal{B} , the combinatorics of the metric circle does not change. If x meets a face of the simplex which corresponds to a coarser partition of the same diagonal arrangement (that is, no diagonals get removed), then again the combinatorics of the metric circle does not change. If x meets a face of the simplex which corresponds to a removal of some diagonals that are not incident to v , then again the combinatorics of the circle does not change. Finally, removal of diagonals that are incident to v means that corresponding lengths $l(d_i)$ tend to zero, and eventually the corresponding edges of the circle collapse.

6. CONTRACTION OF A FREE BOUNDARY COMPONENT

In this section we consider graph complexes \mathcal{C} and \mathcal{B} for the case when some of labeled boundary components of F are *free*, that is, have

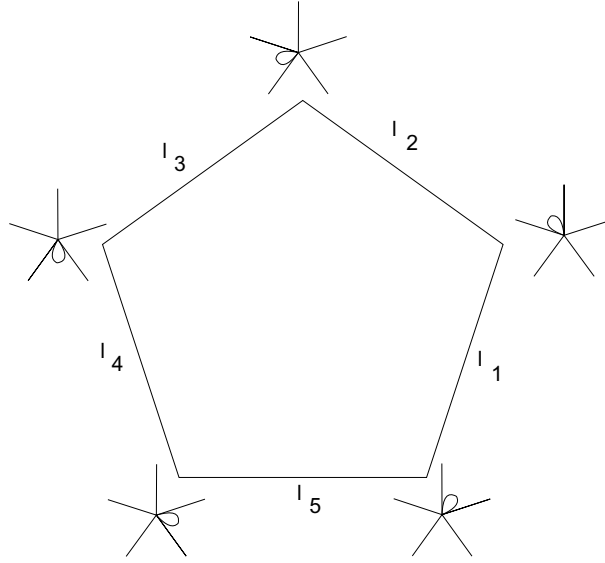


FIGURE 5. The fiber with assigned lengths.

no marked points. One can think of them as *punctures* on the surface. We repeat the same definition of posets C and B as in Section 2, with one necessary exception: in the definition of admissible arrangement instead of saying that "The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of open disks" we say "The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of **punctured** open disks". Note that we allow multiple punctures in a single disk.

It follows from [12], Proposition 12, (also with a different viewpoint, from [11]) that if F is a disc with a number of marked points on its boundary and a number of free boundary components (or punctures) inside the disc, then the associated diagonal complex is a ball with the unique biggest cell, the interior of the ball. This allows to define complexes $\mathcal{C}_{g,b,n,f;n_1,\dots,n_b}$ and $\mathcal{B}_{g,b,n,f;n_1,\dots,n_b}$ analogously to the construction of Section 2. Here f denotes the number of free boundary components.

Example 5. *The complex $\mathcal{C}_{0,1,0,1;1}$ (that is, the complex associated with one-punctured n -gon) is combinatorially isomorphic to the cyclohedron.*

Proof. Given an arrangement of diagonals, cut the polygon by a path connecting the puncture with the boundary of the polygon. We assume that the path does not intersect interiors of the diagonals. Take the copy of the polygon with the same arrangement and with the same cut, and glue the two copies together. We get a (non-punctured) $2n$ -gon together with a centrally symmetric arrangement of diagonals, see

Fig. 6. This construction can be reversed, and therefore establishes a combinatorial isomorphism with the cyclohedron. \square

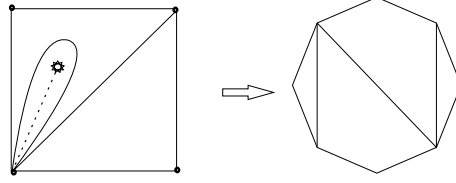


FIGURE 6. Building a symmetric diagonal arrangement.
The dashed line denotes the cut.

Contraction (or removal) of a free boundary component gives rise to a natural forgetful projection defined analogously to forgetful projections from Sections 4 and 5.

We shall need the new surface \overline{F} which is obtained from F by eliminating (or, contracting) all free boundary components and replacing all free marked points by holes.

Theorem 5. *The forgetful projection which removes the free boundary component B_{f+1}^{free}*

$$\pi : \mathcal{B}_{g,b,n,f+1;n_1,\dots,n_b} \rightarrow \mathcal{B}_{g,b,n,f;n_1,\dots,n_b}$$

is a locally trivial bundle over $\mathcal{B}_{g,b,n,f;n_1,\dots,n_b}$ whose fibers are homeomorphic to the (above defined) surface \overline{F} .

Each fiber has a structure of a cell complex compatible with the projection.

Proof. Let us examine the preimage of an inner point x of a cell of $\mathcal{B}_{g,b,n,f;n_1,\dots,n_b}$ labeled by (S_1, \dots, S_r) . The label corresponds to an arrangement $A = \bigcup S_i$ on F with f punctures. We shall show that $\pi^{-1}(x)$ is a cell complex homeomorphic to \overline{F} . Here is its explicit construction: Remove all the free boundary components (or punctures) from F . Now the arrangement A cuts F into some polygons. A *corner* is a vertex with two germs g_1 and g_2 of incident edges, such that there are no other germs between g_1 and g_2 . For each of the corners, we blow up its vertex, that is, replace it by an extra edge, as is shown in Fig. 7. We get a two-dimensional cell complex $\mathcal{F}(A)$ homeomorphic to \overline{F} . Indeed, each free vertex turns to a new boundary component, whereas all the punctures are forgotten.

With a fixed x (and therefore, fixed A), each cell σ of $\mathcal{F}(A)$ gives rise to a cell of the complex $\mathcal{C}_{g,b,n,f+1;n_1,\dots,n_b}$ which intersects $\pi^{-1}(x)$. The corresponding diagonal arrangement $A(x, \sigma)$ on F is described by the following rule (illustrated in Figures 8 and 9):

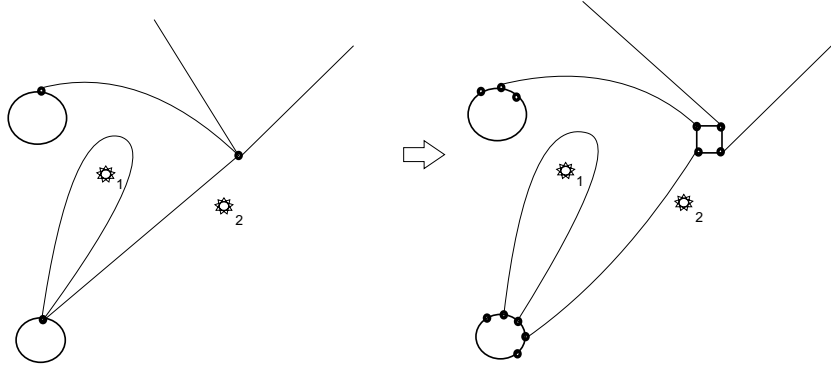


FIGURE 7. Blowing up all the corners. Here we depict a fragment of a bordered surface F . Bold circles denote the boundary components with marked points, small stars denote free boundary components.

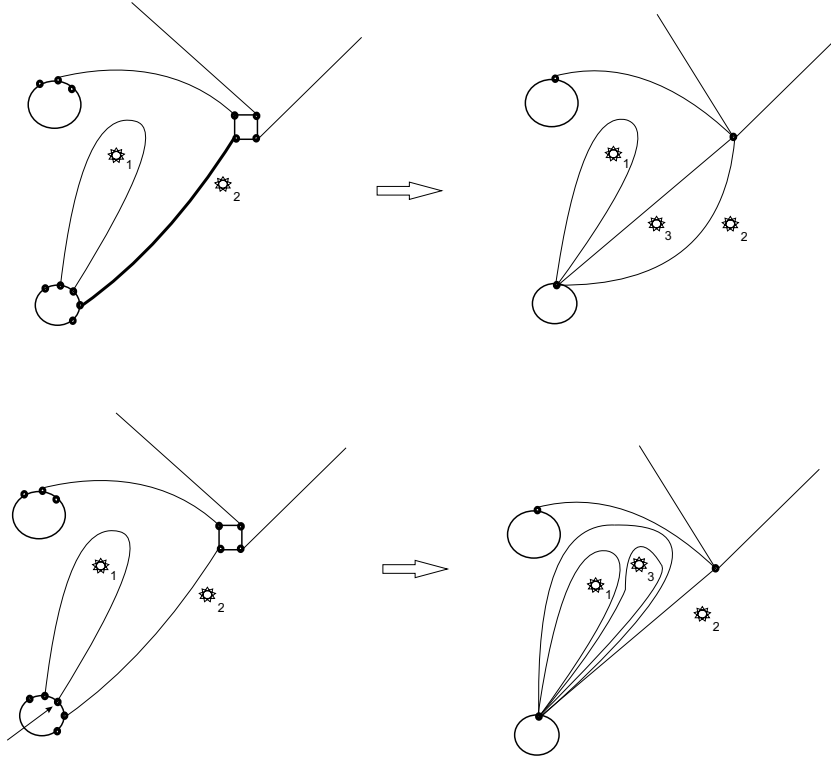


FIGURE 8. Here we depict arrangements that correspond to the bold edge and to the vertex (indicated by the small arrow) of \mathcal{F} .

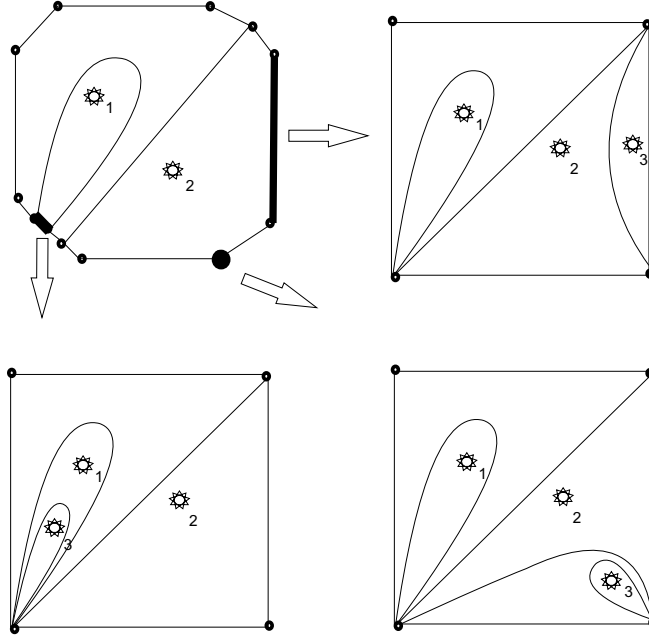


FIGURE 9. One more illustration. Here we depict arrangements that correspond to a vertex and two edges.

- (1) If σ is a 2-dimensional cell, keep the arrangement A and add the new free boundary component B_{f+1}^{free} in the cell σ .
- (2) If the cell σ is an edge e of F , we keep A , add B_{f+1} in a cell σ which is adjacent to e , and also add one more diagonal embracing the puncture which is parallel to e .
- (3) If the cell σ is a blow-up of one of the corners, we keep A , add B_{f+1}^{free} in a cell which is adjacent to e , and also add one loop diagonal embracing the puncture which starts and ends at the corner.
- (4) If the cell is one of the diagonals $d \in A$, duplicate d (keeping the rest of A), and put B_{f+1}^{free} between d and its copy.
- (5) If the cell is one of the new vertices, that is, corresponds to a corner and a vertex of F , we combine either (3) and (2), or (3) and (4), that is, add both a loop and a parallel diagonal.

The proof will be completed once we show that the cells of \mathcal{C} intersected with $\pi^{-1}(x)$ form a regular cell complex on $\pi^{-1}(x)$. To prove this, let us fix a cell Σ of \mathcal{C} which intersects the preimage and analyse the corresponding cells of \mathcal{B} . This means that we set partition on the set of new diagonals. The rules are:

- (1) If the new diagonal d either becomes contractible, or homotopic to an edge after removal of B_{f+1}^{free} , we can put d to any of the sets S_i , and therefore have $(S_1 \cup \{d\}, S_2, \dots, S_r)$, $(S_1, \{d\}, S_2, \dots, S_r)$, $(S_1, S_2 \cup \{d\}, \dots, S_r)$, \dots , $(S_1, S_2, \dots, S_r, \{d\})$ in the preimage.
- (2) If the new diagonal d becomes homotopic to some $d' \in S_i$, we put d either in S_i , or to the right of S_i .

With this understanding we see that if the cell Σ corresponds to adding one new diagonal to A , the intersection $\Sigma \cap \pi^{-1}(x)$ is a combinatorial segment.

If we add two new diagonals, the intersection $\Sigma \cap \pi^{-1}(x)$ is a grid cell complex $r \times (r - i + 2)$, which is a topological disc D^2 .

To see that the bundle is locally trivial, one can think that the blow-ups of the corners have "very small" lengths, proportional to the length of the incident diagonals. \square

Remark. The bundle is not trivializable: observe that each of the tautological bundles L_i (see Definition 2) embeds in it:

$$\begin{array}{ccc}
 L_i & \hookrightarrow & \mathcal{B}_{g,b,n,f+1;n_1,\dots,n_b} \\
 & \searrow & \downarrow \\
 & & \mathcal{B}_{g,b,n,f;n_1,\dots,n_b}
 \end{array}$$

Namely, each fiber of L_i (that is, a combinatorial circle) maps to the new boundary component of \overline{F} that arises from blowing up the corners incident to the free marked point v_i .

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